ellipse and other quantities to a limited degree of accuracy. In the case of a punch with planar base, when the solution of (2.2) is sought in a class of functions having a root singularity on $\partial\Omega$, all the required quantities can be found to any degree of accuracy (in this case, the contact ellipse is assumed to be known).

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AN EFFECTIVE METHOD OF VERIFYING HADAMARD'S CONDITION FOR A NON-LINEARLY ELASTIC COMPRESSIBLE MEDIUM†

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A new effective criterion is proposed for the validity of Hadamard's condition in a non-linearly elastic compressible body. The verification of Hadamard's condition reduces to analysing a simply structured system of inequalities, so that its validity can be investigated by analytical means, using the same technique for all compressible materials.

INTRODUCTION

IT HAS been shown [1] that for an isotropic incompressible material Hadamard's condition, according to which the velocities of propagation of plane waves of small amplitude in a uniformly stressed elastic medium must be real [2, 3], is equivalent to a system of nine elementary inequalities.

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For a compressible medium, however, though a method that should lead to elementary inequalities equivalent to Hadamard's condition has been indicated [1], the structure of some of these inequalities is so complicated that no finite system has actually been presented. Nevertheless, following the technique developed in [1], one can derive 12 elementary inequalities

$$\alpha_k \geqslant 0, \quad \beta_k \geqslant 0, \quad \gamma_k^{\pm} + \sqrt{\beta_i \beta_j} \geqslant 0 \tag{0.1}$$

in which the parameters α_k, β_k, $\gamma_k^{\pm}$ $(k = 1, 2, 3)$ are expressed in terms of the elastic potential Π, and four implications of the form

$$\bigwedge_{n=1}^{6} (\delta_{mn} \geqslant 0) \Rightarrow \Delta_m \geqslant 0 \quad (m = 0, 1, 2, 3) \tag{0.2}$$

The indices i, j, k in (0.1) constitute any permutation of the numbers 1, 2, 3. The symbol $\wedge$ denotes conjunction of statements, while δ_{mn}, Δ_m $(m = 0, 1, 2, 3;\ n = 1, \ldots, 6)$ denote the cofactors and determinants, respectively, of the matrices

$$a_0 = \begin{Vmatrix} \beta_1 & \gamma_3^+ & \gamma_2^+ \\ \gamma_3^+ & \beta_2 & \gamma_1^+ \\ \gamma_2^+ & \gamma_1^+ & \beta_3 \end{Vmatrix}, \quad a_1 = \begin{Vmatrix} \beta_1 & \gamma_3^- & \gamma_2^- \\ \gamma_3^- & \beta_2 & \gamma_1^+ \\ \gamma_2^- & \gamma_1^+ & \beta_3 \end{Vmatrix}$$
$$a_2 = \begin{Vmatrix} \beta_1 & \gamma_3^- & \gamma_2^+ \\ \gamma_3^- & \beta_2 & \gamma_1^- \\ \gamma_2^+ & \gamma_1^- & \beta_3 \end{Vmatrix}, \quad a_3 = \begin{Vmatrix} \beta_1 & \gamma_3^+ & \gamma_2^- \\ \gamma_3^+ & \beta_2 & \gamma_1^- \\ \gamma_2^- & \gamma_1^- & \beta_3 \end{Vmatrix} \tag{0.3}$$

Inequalities (0.1) are identical in structure to the corresponding inequalities for an incompressible material and are generally not amenable to further simplification. On the other hand, the implications (0.2) are rather complicated in structure; to verify the validity of each of them one has to compute seven determinants, six of second-order and one of third-order—a fairly difficult task.

The device proposed in [1] for deriving elementary inequalities equivalent to Hadamard's condition relies on a criterion, formulated by Gurvich and Lur'ye, for a quadratic form of N variables (where N is any natural number) to be partially positive semidefinite. Below we shall propose a formulation of the criterion, different from that in [1], for the special case $N = 3$ (Theorem 1); using this criterion one can obtain a more simply structured system of elementary inequalities. This gives a substantial reduction in the amount of computations necessary to verify Hadamard's condition. It will be proved that when $N = 3$ the Gurvich–Lur'ye criterion is equivalent to our Theorem 2. An effective sufficient condition will be established for Hadamard's condition to be valid (Theorem 3). A mechanical interpretation of the individual inequalities of system (0.1) will be presented. Examples of compressible elastic materials will be considered. The method may be used to obtain effective criteria for the equilibrium equations of a non-linearly elastic medium to be strongly elliptic [2, 4] and for an isotropic compressible material to be positively longitudinally elastic [3].

1. THE CRITERION FOR A QUADRATIC FORM OF THREE VARIABLES TO BE PARTIALLY POSITIVE SEMIDEFINITE

Consider a quadratic form

$$L(x) = \sum_{i,j=1}^{3} a_{ij} x_i x_j \tag{1.1}$$

We shall say that $L(x)$ is partially positive semidefinite if it is non-negative for any $x_1 \geqslant 0$, $x_2 \geqslant 0$, $x_3 \geqslant 0$ [1].

Theorem 1. The quadratic form (1.1) is partially positive semidefinite if and only if, for any permutation i, j, k of the indices 1, 2, 3,

$$a_{kk} \geqslant 0, \quad a_{ij} + \sqrt{a_{ii} a_{jj}} \geqslant 0 \tag{1.2}$$

$$(a_{ki}<0)\wedge(a_{kj}<0)\Rightarrow a_{kk}a_{ij}-a_{ki}a_{kj}+ + (a_{kk}a_{ii}-a_{ki}^2)^{1/2}(a_{kk}a_{jj}-a_{kj}^2)^{1/2}\geqslant 0 \tag{1.3}$$

Proof. The necessity of conditions (1.2) is obvious, so we need only prove the necessity of the implication (1.3). Indeed, assuming that $a_{ki}<0$, $a_{kj}<0$, fix arbitrary values of $x_i\geqslant 0$, $x_j\geqslant 0$ and consider the behaviour of $L(x)$ as a function of x_k. By conditions (1.2) and our assumption that a_{ki}, a_{kj} are negative, we have $a_{kk}>0$. Therefore $L(x)$ is a quadratic trinomial in x_k with positive leading coefficient, which, as can be checked reaches a minimum at the point

$$x_k^0=-a_{kk}^{-1}(a_{ki}x_i+a_{kj}x_j)\geqslant 0 \tag{1.4}$$

The minimum value of $L(x)$ is obtained by substituting x_k^0 into (1.1):

$$\min_{x_k} L(x) = a_{kk}^{-1}[(a_{kk}a_{ii}-a_{ki}^2)x_i^2+(a_{kk}a_{jj}-a_{kj}^2)x_j^2+ + 2(a_{kk}a_{ij}-a_{ki}a_{kj})x_ix_j] \tag{1.5}$$

Since $L(x)$ is positive semidefinite and $x_k^0\geqslant 0$, the quadratic form on the right of (1.5) must be non-negative for all $x_i\geqslant 0$, $x_j\geqslant 0$, and this implies the conclusion of the implication (1.3).

We will now prove sufficiency. Clearly, inequalities (1.2) guarantee that $L(x)$ is partially positive semidefinite in the coordinate planes $x_m=0$ ($m=1, 2, 3$). Discarding the trivial case in which some of the diagonal elements a_{11}, a_{22}, a_{33} vanish, we may assume that $a_{mm}>0$ ($m=1, 2, 3$). There are two possibilities: either two of the coefficients a_{12}, a_{23}, a_{31} are non-negative or two of them are negative. The first case causes no difficulties. In the second case there is a permutation i, j, k such that $a_{ki}<0$, $a_{kj}<0$ and, as is readily checked, $L(x)$ attains a minimum at the point x_k^0 defined by (1.4). By (1.5), the second condition of (1.2) and (1.3) imply that $\min_{x_k} L(x)$ is non-negative for $x_i\geqslant 0$, $x_j\geqslant 0$, so that $L(x)$ is indeed positive semidefinite. This proves the theorem.

Theorem 2. In the special case of a form in three variables, the Gurvich–Lur'ye criterion [1] is equivalent to the criterion formulated in Theorem 2.

Proof. Let a be the matrix of the form $L(x)$ and A the adjoint matrix of a, that is, the matrix whose elements A_{ij} are the cofactors of the a_{ij}s. According to the Gurvich–Lur'ye criterion, which is applicable for arbitrary N (N being the order of the form), a quadratic form $L(x)$ is partially positive semidefinite if and only if (a) it possesses that property over each of the coordinate planes $x_m=0$ ($m=1, \ldots, N$); (b) if the elements of the adjoint matrix A are non-negative, then so is the determinant of a.

Assuming that inequalities (1.2) are satisfied, we shall prove that the following assertions are equivalent:

(1) the implications (1.3) hold for any permutation i, j, k of the numbers 1, 2, 3;

(2) if all the elements of the adjoint matrix A are non-negative, then $\det a\geqslant 0$, i.e.

$$\bigwedge_{m,\,n=1}^{3}(A_{mn}\geqslant 0)\Rightarrow \det a\geqslant 0 \tag{1.6}$$

To that end, we shall show that if one of these assertions is false, then so is the other. Suppose, for exaple, that assertion 1 is false. Then there is a permutation i, j, k of 1, 2, 3, such that

$$a_{ki}<0,\quad a_{kj}<0 \tag{1.7}$$

$$a_{kk}a_{ij}-a_{ki}a_{kj}+(a_{kk}a_{ii}-a_{ki}^2)^{1/2}(a_{kk}a_{jj}-a_{kj}^2)^{1/2}\geqslant 0 \tag{1.8}$$

It will then follow from (1.2) and (1.7) that $a_{kk}>0$.

It can be shown that all the elements A_{mn} (m, $n=1, 2, 3$) of the adjoint matrix A are non-negative.

In fact, for A_{ii}, A_{jj} this follows from (1.7) and the second inequality of (1.2); and for A_{ij} it follows from (1.6), since inequality (1.8) can be written in the form

$$A_{ij} > \sqrt{A_{ii}A_{jj}} \tag{1.9}$$

Before considering the three remaining elements A_{ki}, A_{kj} and A_{kk}, let us determine the sign of $\det a$. To that end we use the identity

$$\det a = a_{kk}^{-1}(A_{ii}A_{jj} - A_{ij}^2) \tag{1.10}$$

Since $a_{kk} > 0$, it follows by (1.9) that

$$\det a < 0 \tag{1.11}$$

Writing down the explicit expressions for A_{ki}, A_{kj}, A_{kk}, the readers can convince themselves that when $a_{ij} < 0$ they are all non-negative, by (1.2) and (1.7). But if $a_{ij} \geqslant 0$, it will again follow from the formula

$$\det a = a_{ii}A_{ii} + a_{ij}A_{ij} + a_{ik}A_{ik} \tag{1.12}$$

and from inequality (1.11) that $A_{ik} \geqslant 0$ (and similarly for A_{jk}). Further, using the inequality $A_{ij} \geqslant 0$, one obtains an upper bound for a_{ij}; applying this bound together with (1.7) and (1.2), we see that in this case too $A_{kk} \geqslant 0$.

Since all the elements A_{mn} $(m,\ n = 1,\ 2,\ 3)$ are non-negative while at the same time inequality (1.11) is valid we have a contradiction to (1.6), as required.

Let us assume now that assertion 2 is false. This means that $A_{mn} \geqslant 0$ $(m,\ n = 1,\ 2,\ 3)$. We shall show that because of this there exists a permutation i, j, k of 1, 2, 3 such that conditions (1.7), (1.8) are valid, i.e. assertion 1 is also false. Indeed, under our assumptions, using formulas similar to (1.12), one can show that each row of a contains at least one negative off-diagonal element. But then there must be a row in which both off-diagonal elements are negative, say row k. Then conditions (1.7) are satisfied for indices i, j chosen arbitrarily to complete k to a permutation i, j, k of 1, 2, 3. As to (1.8), it is equivalent to inequality (1.9) and is thus true thanks to (1.10) and (1.11).

We have thus proved the theorem. Simultaneously, we have found a fairly simple proof of the Gurvich–Lur'ye criterion for $N = 3$.

2. THE SYSTEM OF ELEMENTARY INEQUALITIES EQUIVALENT TO HADAMARD'S CONDITION

We will use the following notation (i, j, k is an arbitrary permutation of the indices 1, 2, 3):

$$\alpha_k = \frac{\Pi_i v_i - \Pi_j v_j}{v_i^2 - v_j^2}, \quad \beta_k = \Pi_{kk}, \quad \gamma_k^{\pm} = \pm\Pi_{ij} + \frac{\Pi_i \mp \Pi_j}{v_i \mp v_j},$$
$$\Pi_i \equiv \frac{\partial\Pi}{\partial v_i}, \quad \Pi_{ij} \equiv \frac{\partial^2\Pi}{\partial v_i \partial v_j} \tag{2.1}$$

We will assume that the specific potential energy of the deformation Π is a twice continuously differential function of the principal dilatations v_1, v_2 and v_3 [2, 3], also known as the principal multiplicities of the elongations. For a compressible material, Hadamard's condition is equivalent [1] to the condition that the four quadratic forms in three variables with matrices a_0, a_1, a_2, a_3 [see (0.3)] are partially positive semidefinite. Therefore, Theorem 1 yields the following system of elementary inequalities:

$$\alpha_k \geqslant 0, \quad \beta_k \geqslant 0 \tag{2.2}$$

$$\gamma_k^{\pm} + \sqrt{\beta_i\beta_j} \geqslant 0 \tag{2.3}$$

$$(\gamma_i^m < 0) \wedge (\gamma_j^n < 0) \Rightarrow \beta_k\gamma_k^{mn} - \gamma_i^m\gamma_j^n + [\beta_k\beta_j - (\gamma_i^m)^2]^{1/2}[\beta_k\beta_i - (\gamma_j^n)^2]^{1/2} \geqslant 0 \tag{2.4}$$

where i, j, k is an arbitrary permutation of 1, 2, 3. The symbols m, n in (2.4) take values plus and minus, and their product mn is defined by the usual rule for multiplication of the numbers $+1$, -1, i.e. mn is plus if m, n are the same, minus otherwise. Condition (2.4) must hold for any choice of signs (2.4).

The implication (2.4) is clearly much simpler than (0.2). As shown by actual examples, it may frequently be checked by entirely analytical means. It is also worth stressing that of the twelve implications (2.4) corresponding to all permutations i, j, k of 1, 2, 3 and all sign combinations m, n, at least nine will always be valid if inequalities (2.2), (2.3) are valid.

Indeed by (2.1), the parameters γ_k^+, γ_k^- and α_k satisfy the condition

$$\gamma_k^+ + \gamma_k^- = 2\alpha_k \tag{2.5}$$

which shows that γ_k^+, γ_k^- cannot both be negative if $\alpha_k \geqslant 0$.

Thus, of the six quantities $\gamma_k^\pm$ ($k = 1, 2, 3$), at most three are negative, so that in the "worst possible" case one has to check only three implications (2.4). The nine others will be valid automatically, since their premises are false. It can also be shown that the "worst possible" scenario involves three negative coefficients of the form $\gamma_i^+, \gamma_j^+, \gamma_k^-$ or $\gamma_i^-, \gamma_j^-, \gamma_k^-$, while in all other cases it is either necessary to check only one implication (2.4) (at least two negative coefficients) or they are all surely true (one negative coefficient or none). Consequently, the criterion established in Theorem 1 not only considerably simplifies the structure of the implications (0.2) but reduces the number of implications to be checked.

Thus, for a compressible material, a system of elementary inequalities equivalent to Hadamard's condition will generally consist of 12 unconditional inequalities (2.2), (2.3) and three conditional ones of type (2.4), i.e. all in all 15 inequalities (as against nine for an incompressible material). True, using (2.5) one can show that the pair of inequalities (2.3) may be replaced by their product

$$(\gamma_k^+ + \sqrt{\beta_i\beta_j})(\gamma_k^- + \sqrt{\beta_i\beta_j}) \geqslant 0 \tag{2.6}$$

or

$$\beta_i\beta_j + 2\alpha_k\sqrt{\beta_i\beta_j} + \gamma_k^+\gamma_k^- \geqslant 0 \tag{2.7}$$

but by the accepted convention [1] inequalities (2.7) cannot be considered elementary, since each of them splits into two simpler ones.

It is well known [2, 3] that inequalities (2.2) admit of a simple physical interpretation (in both the static and acoustic contexts). In particular, the conditions $a_k \geqslant 0$ are a weakened version of the Baker–Ericksen inequalities [5, 3], representing the statement that if one compares two principal expansions in a given element of the material, the larger of them corresponds to a principal stress which is at least as large.

It turns out that inequalities (2.3) also have a clear-cut mechanical meaning. Assuming that inequalities (2.2) are valid and using the usual representation for the components of the acoustic tensor [2, 3] in the principal axes of Finger's measure of deformation, one can show that in a homogeneous stressed elastic medium the squared velocities of plane waves polarized in the principal plane $x_k = 0$ and travelling in the direction of the normal **n** with components

$$n_i^2 = \sqrt{\beta_j}v_j^2\theta, \quad n_j^2 = \sqrt{\beta_i}v_i^2\theta, \quad n_k^2 = 0 \tag{2.8}$$
$$\theta = (\sqrt{\beta_i}v_i^2 + \sqrt{\beta_j}v_j^2)^{-1}$$

are determined by the relations

$$\rho_0 c_1^2 = K_1(\gamma_k^+ + \sqrt{\beta_i\beta_j})(\gamma_k^- + \sqrt{\beta_i\beta_j}), \quad \rho_0 c_2^2 = K_2 \tag{2.9}$$
$$K_1 = 2\sqrt{\beta_i\beta_j}v_i^2v_j^2\theta M^{-1}, \quad K_2 = {}^1/_2 v_i^2 v_j^2 \theta M$$
$$M = (\sqrt{\beta_i} + \sqrt{\beta_j})(\alpha_k + \sqrt{\beta_i\beta_j}) + [(\sqrt{\beta_i} - \sqrt{\beta_j})^2 \times$$
$$\times (\alpha_k + \sqrt{\beta_i\beta_j})^2 + (\gamma_k^+ - \gamma_k^-)^2 \sqrt{\beta_i\beta_j}]^{1/2}$$

where ρ_0 is the density of the material in the undeformed state; c_1 and c_2 are the wave velocities. Clearly, a sufficient condition for c_2 to be real is the validity of inequalities (2.2); but for c_1 to be real, we need, besides (2.2), also inequalities (2.6) [or (2.3)].

In (2.8) and (2.9) it is assumed that

$$(\alpha_k^2+\beta_i^2)(\alpha_k^2+\beta_j^2)(\beta_i^2+\beta_j^2)\neq 0 \tag{2.10}$$

If $\beta_i = \beta_j = 0$ but $\alpha_k \neq 0$, then formulas (2.9) are valid for any normal $\mathbf{n}$ in the plane $x_k = 0$, but the formulas for K_1, K_2 and M become

$$K_1=2v_i^2v_j^2n_i^2n_j^2M^{-1}, \quad K_2={}^1/_2M$$

$$M=\alpha_k(v_i^2n_i^2+v_j^2n_j^2)+[\alpha_k^2(v_i^2n_i^2-v_j^2n_j^2)^2+(\gamma_k^+-\gamma_k^-)^2\times \times v_i^2v_j^2n_i^2n_j^2]^{1/2}$$

In all other cases in which (2.10) is false, we have $c_1 = c_2 = 0$, $\gamma_k^{\pm} + \sqrt{\beta_i\beta_j} = 0$, i.e. formulas (2.9) may again be considered valid, if we put $K_2 = 0$. In that case the corresponding direction $\mathbf{n}$ is either defined by (2.8) (if the latter are meaningful) or any direction in the plane $x_k = 0$ (otherwise).

Thus, inequalities (2.6) [or (2.3)] express the condition that c_1 is real.

Inequalities (2.2) and (2.3) taken together are necessary and sufficient for Hadamard's condition to hold in the principal planes of the Finger measure of deformation. As for the implications (2.4), they should be regarded as additional constraints, which guarantee that the elastic wave velocities will be real not only in the principal planes but in any direction $\mathbf{n}$.

Our results up to this point imply the following test for the validity of Hadamard's condition.

Theorem 3. If v_1, v_2 and v_3 are given and for any permutation i, j, k of the indices 1, 2, 3 and any combination m, n of the signs plus, minus inequalities (2.2) and (2.3) are true and moreover

$$(\gamma_i^m<0)\wedge(\gamma_j^n<0)\Rightarrow\beta_k\gamma_k^{mn}-\gamma_i^m\gamma_j^n\geqslant 0 \tag{2.11}$$

then the material under consideration satisfies Hadamard's condition at the point (v_1, v_2, v_3) of the dilatation space.

The proof is obvious and will therefore be omitted.

Remarks. 1. If one replaces the symbol $\geqslant$ throughout (2.2)–(2.4) by $>$, the result is a system of elementary inequalities equivalent to the condition that the equilibrium equations of a non-linearly elastic medium be elliptic [2, 3]. A similar version of Theorem 3 is also valid.

2. Using Theorem 1, one can formulate an effective criterion for an isotropic compressible material to be positively longitudinally elastic [3]; we shall not dwell on that here.

3. VERIFICATION OF HADAMARD'S CONDITION

We will now consider a few examples of the verification of Hadamard's condition by the technique outlined in Sec. 2.

Compressible Mooney–Rivlin material (or Hadamard material) [2]

The specific potential energy of the deformation is defined by

$$\Pi=c_1I_1+c_2I_2+f(I_3) \quad (c_1>0, \ c_2>0) \tag{3.1}$$

$$I_1=v_1^2+v_2^2+v_3^2, \quad I_2=v_1^2v_2^2+v_2^2v_3^2+v_3^2v_1^2, \quad I_3=v_1^2v_2^2v_3^2$$

where I_1, I_2 and I_3 are the principal invariants of the Finger deformation measure, f is a twice continuously differentiable function of the third invariant I_3 and c_1 and c_2 are constants. If the reference configuration is identical with the natural state and the latter is given the value $\Pi = 0$, then f must also satisfy the condition

$$f(1)+3(c_1+c_2)=0, \quad f'(1)+c_1+2c_2=0 \tag{3.2}$$

where the prime means differentiation with respect to I_3. It is easy to see that Π will be positive if and only if

$$3(c_1 I_3^{1/3}+c_2 I_3^{2/3})+f(I_3)>0 \quad (I_3\neq 1) \tag{3.3}$$

Indeed the necessity follows from an examination of the potential (3.1) on the straight line $v_1 = v_2 = v_3$ in the dilatation space, while the sufficiency follows by using the obvious bounds

$$I_1\geqslant 3I_3^{1/3}, \quad I_2\geqslant 3I_3^{2/3}$$

and the fact that $c_1>0$, $c_2>0$.

Putting $c_3\equiv f'+2I_3 f''$, we deduce from (2.1) and (3.1) that

$$\begin{gathered}\alpha_k=2(c_1+c_2 v_k^2), \quad \beta_k=2[c_1+c_2(v_i^2+v_j^2)+c_3 v_i^2 v_j^2] \\ \gamma_k^{\pm}=2(c_1+c_2 v_k^2)\pm 2v_i v_j(c_2+c_3 v_k^2)\end{gathered} \tag{3.4}$$

where i, j, k is an arbitrary permutation of 1, 2, 3. Conditions 2.2 become

$$c_1+c_2 v_k^2\geqslant 0, \quad c_1+c_2(v_i^2+v_j^2)+c_3 v_i^2 v_j^2\geqslant 0 \tag{3.5}$$

We shall show that inequalities (3.5) imply that the quantities γ_k^+ ($k = 1, 2, 3$) are non-negative. Indeed, this is obvious if $c_2+c_3 v_k^2\geqslant 0$, while if $c_3+c_3 v_k^2<0$ it follows from the identity

$$2\gamma_k^+=\beta_i+\beta_j-2(v_i-v_j)^2(c_2+c_3 v_k^2) \tag{3.6}$$

which in turn is easily verified using (3.4).

Thus, the inequalities

$$\gamma_k^+ + \sqrt{\beta_i\beta_j}\geqslant 0 \tag{3.7}$$

will always be valid if conditions (3.5) are satisfied.

We will now verify that

$$\gamma_k^- + \sqrt{\beta_i\beta_j}\geqslant 0 \tag{3.8}$$

If $c_2+c_2 v_k^2\leqslant 0$, then $\gamma_k^->0$ and (3.8) is valid. Let $c_2+c_3 v_k^2>0$. Then by (3.4) and (3.5),

$$\beta_i\geqslant 2v_j^2(c_2+c_3 v_k^2), \quad \beta_j\geqslant 2v_i^2(c_2+c_3 v_k^2)$$

which implies that

$$\gamma_k^- + \sqrt{\beta_i\beta_j}\geqslant 2(c_1+c_2 v_k^2)\geqslant 0$$

i.e. conditions (3.8) are valid in any case.

Finally, let us consider the implication (2.4). As shown previously, only the quantities γ_k^- ($k = 1, 2, 3$) may be negative. Suppose that $\gamma_i^-<0$, $\gamma_j^-<0$. Then by (3.4) we must have

$$c_2+c_3 v_i^2>0, \quad c_2+c_3 v_j^2>0 \tag{3.9}$$

A direct check will now convince the readers of the truth of the identity

$$\begin{gathered}\beta_k\gamma_k^+-\gamma_i^+\gamma_j^-=4(v_k+v_i)(v_k+v_j)\times \\ \times[(c_1+c_2 v_i v_j)(c_2+c_3 v_i v_j)+c_2^2(v_i-v_j)^2]\end{gathered} \tag{3.10}$$

in which the right-hand side is non-negative, because by (3.9), irrespective of the sign of c_3, we have $c_2+c_3 v_i v_j>0$.

Thus, by Theorem 3, inequalities (3.5) are not only necessary but also sufficient for Hadamard's condition to hold. Incidentally, our proof of this statement made no use of the restrictions $c_1>0$, $c_2>0$. If we assume that they are valid, then the only inequalities remaining in (3.5) are

$$c_1+c_2(v_i^2+v_j^2)+c_3 v_i^2 v_j^2\geqslant 0 \tag{3.11}$$

This condition is clearly valid for all v_1, v_2, v_3 if and only if

$$c_3\equiv f'+2I_3 f''\geqslant 0 \tag{3.12}$$

Consequently, a compressible Mooney–Rivlin material will satisfy Hadamard's condition for arbitrary deformations if and only if condition (3.12) is satisfied.

As an example, consider the three-constant potential

$$\Pi = c_1(I_1-3)+c_2(I_2-3)+c_0(I_3-1)+(c_1+2c_2+c_0)(I_3^{-1}-1)$$

$$(c_0,\ c_1,\ c_2>0)$$

It is readily seen that conditions (3.2) and (3.3) hold for this potential. Moreover,

$$f(I_3)=c_0(I_3-1)+(c_1+2c_2+c_0)(I_3^{-1}-1)-3(c_1+c_2)$$

$$c_3=c_0+\frac{3(c_1+2c_2+c_0)}{I_3^2}>0$$

i.e. equality (3.12) is valid.

Signorini material [2]

We will confine our attention to the simplified Signorini model, which is the potential

$$\Pi={}^1/_8 J_3^{-1/2}[(9\lambda+5\mu)-2(3\lambda+\mu)J_1+(\lambda+\mu)J_1^2]-\mu \tag{3.13}$$

$$J_1=v_1^{-2}+v_2^{-2}+v_3^{-2},\quad J_3=v_1^{-2}v_2^{-2}v_3^{-2};\quad \lambda,\ \mu=\text{const}$$

where J_1 and J_3 are the first and third principal invariants of the Almansi measure of deformation [2, 3]. The sufficient conditions for Π to be positive are [2]

$$\mu>0,\quad 9\lambda+5\mu>0 \tag{3.14}$$

Without dwelling on the proof, we note that these conditions are not only sufficient but also necessary.

Using (2.1) and (3.13), we obtain

$$\alpha_k={}^1/_2 v_i^{-1}v_j^{-1}v_k[(\lambda+\mu)J_1-(3\lambda+\mu)]$$

$$\beta_k={}^1/_2 v_iv_jv_k^{-3}[(\lambda+\mu)J_1-(3\lambda+\mu)+2(\lambda+\mu)v_k^{-2}] \tag{3.15}$$

$$\gamma_k^{\pm}={}^1/_2 v_i^{-1}v_j^{-1}v_k[(\lambda+\mu)J_1-(3\lambda+\mu)\pm2(\lambda+\mu)v_i^{-1}v_j^{-1}]$$

Using these formulas, we reduce inequalities (2.4) to the form

$$(\lambda+\mu)J_1-(3\lambda+\mu)\geqslant 0 \tag{3.16}$$

$$(\lambda+\mu)J_1-(3\lambda+\mu)+2(\lambda+\mu)v_k^{-2}\geqslant 0$$

It is clear that conditions (3.16) are sufficient for the quantities γ_k^+ $(k=1,\ 2,\ 3)$ to be non-negative. Thus inequalities (3.7) are satisfied. The proof of (3.8) is exactly the same. Thus, conditions (2.3) are in this case corollaries of (2.2).

Now, by (3.15),

$$\beta_k\gamma_k^+-\gamma_i^-\gamma_j^-={}^1/_2 v_k^{-2}(v_k^{-1}+v_i^{-1})(v_k^{-1}+v_j^{-1})(\lambda+\mu)\times \times[(\lambda+\mu)J_1-(3\lambda+\mu)] \tag{3.17}$$

If $\gamma_i^-<0$, $\gamma_j^-<0$, then necessarily $\lambda+\mu>0$; thus it follows from (3.17) that the implications (2.4) are always valid if inequalities (3.16) are valid.

Thus, Hadamard's condition for the case of the simplified Signorini law (3.13) reduces to inequalities (3.16). Incidentally, if the necessary and sufficient conditions (3.14) are satisfied, the second inequality of (3.16) follows from the first, because in that situation $\lambda+\mu>0$. Thus all that remains in (3.16) is a single inequality. For the model (3.13), therefore [assuming the restrictions (3.14)], we see that Hadamard's condition is valid for any deformations if and only if

$$3\lambda+\mu\leqslant 0 \tag{3.18}$$

Note that inequality (3.18) is compatible with (3.14), as may be verified by setting, say, $\lambda = -4$ and $\mu = 9$.

Similar examinations may be made for other examples of isotropic compressible elastic materials. In particular, the method of Sec. 2 reproduces previously known results in connection with the verification of Hadamard's condition (for a semilinear John material [2], special cases of Blatz–Ko materials [6], or an elastic Euler fluid [3]), obtained by applying Sylvester's criterion to the matrix of components of the acoustic tensor or by various artificial devices [1, 2, 4, 7, 8]. It should also be noted that in all the models just listed, as in (3.1) and (3.13), and various others, the premises of Theorem 3 are valid; thus the range of applicability of that theorem is fairly wide.

Analysis of specific examples indicates that the technique outlined in Sec. 2 for verifying Hadamard's condition is quite effective. Its advantage over other methods (such as the use of Sylvester's criterion [2, 4] or different kinds of artificial devices [1, 8]) is, on the one hand, the fact that the operations to be applied are the same for all compressible materials; on the other, conditions (2.2)–(2.4) involve no additional parameters. In addition, the comparative simplicity of inequalities (2.2)–(2.4) generally makes it possible to check the validity of Hadamard's condition by analytical means. This can be done, for example, with such models as Murnaghan materials [2] and the general case of Signorini [2] and Blatz–Ko [2, 6] materials. When other methods are effective, the method of this paper is no more laborious and leads just as rapidly to the desired result.

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